NON-EUCLIDEAN GEOMETRY

I. Introduction

Euclidean geometry, the kind most of us studied in high school, works very well for small jobs like designing a building or surveying a city lot. Similarly, the theory that the Earth is flat is quite satisfactory if you stay close to home. But if you assume that the Earth is flat and try to map something big, say Lake Michigan, you will find that something is wrong; when you have gone all the way around the lake, and are back where you started, you will have to fudge your measurements to make the shoreline meet at a point. The nature of the difficulty becomes obvious if you leave the Earth and take a trip to the Moon.

If you try to map the universe, you will encounter more serious difficulties, and there will be no way to make them appear obvious; you cannot get outside the universe and look back to see whether it is "flat".

The best we can do for a start is to illustrate the problems of geometry on the cosmic scale by trying a few things on the surfaces of lab-sized spheres.

II. Constructions, Observations, Questions

A. Review of Euclidean Geometry

On a flat sheet of 8 1/2" x 11" paper draw two straight parallel lines several inches apart, two straight lines that cross at a point, and a large equilateral triangle. Use a compass to locate the corners of the triangle.

The parallel lines are everywhere the same distance apart. They do not cross. Of course, that is what you mean by parallel lines.

The lines that cross only cross at one point. As you move along one of the lines away from the crossing point, you get farther from the other line and farther from where you started. Of course!

Measure the angle between two sides of the triangle with a protractor. Maybe you remember what the answer should be, but measure it anyway.

B. Almost Euclidean Geometry

On one of the spheres provided in the lab, lay out an area about the size of the sheet of paper used in Part A. If you ignore everything outside the area, you could draw the same figures you did before using a flexible plastic ruler and a compass or protractor, and everything would be the same as before— almost. (The Flat Earth theory is nearly OK if you stay close to home.) Let us try it. Draw the parallel lines (several inches apart) and the crossing lines. Near the boundary of the area, draw one side of a triangle the same size as the one on the flat sheet. Now use the protractor to measure the same angle you found in Part A. Draw that side of the triangle the same length as the first. Again use the protractor to find the direction for the third side.

Draw that third side the same length. Did you get back where you started? Did the triangle close? If not, make a sketch on the flat sheet (your map) showing in what way the figure fails to be a triangle.

Now use a compass to lay out a large equilateral triangle inside the area. Join the points using the flexible ruler. This time you are guaranteed to get a figure that is closed, equilateral, and equiangular. Measure the angles with the protractor. What do you get?

C. Non-Euclidean Geometry

Now erase the boundaries of the area on the sphere.

1. Extend the (well-separated) parallel lines, keeping the flexible ruler flat on the surface. Is anything different? Do the lines stay the same distance apart? Do they ever cross? If so, where? How often?
2. Extend the lines that cross. As you move away from the crossing point, do the lines keep getting farther away from each other? Do they ever become parallel? Do they cross again? Where? Are they different from parallel lines?
3. Keeping the flexible ruler flat on the surface, draw a line all the way around the sphere. If you do it carefully, you will get back where you started. You have drawn a "great circle." Notice that if you were walking along a great circle, you would be going straight: That is, you would be turning neither to right nor to left. On the Earth, the equator is an example of a great circle. What is another example?

Now using the compass, mark off a curve that is everywhere the same distance from the great circle, and another curve parallel to that, etc., from the great circle. On the Earth, starting from the equator, these would be lines of constant latitude. Notice that to walk along a line of latitude, you must continuously turn to the right or left. The farther you are from a great circle, the faster you have to turn. Imagine walking in a 10 foot diameter circle around the North Pole.

Pick two points on a great circle. Stretch a string between them. Does the string follow the curve? Stretch a string between two points on a line of longitude. Does the string follow the line? Stretch a string between two points at the same latitude (but not on the equator). Does the string follow a line of constant latitude on the globe? On a globe find the shortest route from Chicago to Tokyo.

Draw a big triangle using the compass system. Now how big are the angles? Describe the biggest triangle you can draw on the sphere. How big are its angles?

D. The Expanding Universe

This part is about the expanding universe, not the big bang. The balloon you will use should not be over-inflated or punctured. Do not use a pointed compass.

On the uniflated balloon draw an equilateral triangle, draw a scale marked off in arbitrary units (cm would be OK), and draw several dots labeled A, B, C, etc. Inflate the balloon until the distance AB has doubled. What has happened to the triangle? Is it still equilateral? Equiangular? Have the angles changed? Has the size of the triangle changed as measured with the plastic ruler? Has it changed as compared with the scale on the balloon?

What has happened to the distance between C and D? Are there any two dots for which you could not say the same thing?

If the universe and everything in it is twice as big today as it was yesterday, could we detect the change? (Assume that all measuring sticks are bigger, and that all clocks are slower in proportion to the change in size.)

Inflate the balloon some more. As it expands, what happens to any two points? (Now we are back to measuring sticks and clocks that stay fixed while galaxies recede.) Is there any special, central point?

If at some instant you measure the distance, D, between two points, and the speed V at which they are receding, how could you guess how long the balloon has been expanding? The laboratory assistant will discuss how to measure the speed at which other galaxies recede from ours.

E. Straight Lines

One way to explain the differences between Part A and C is to say that they involve different concepts of a straight line. In each case, we could use a stretched string, but in C the string was bent by the sphere and was not "really" straight. So what is really straight?

How about sighting along the string? Stick a straight wire into water at an angle. It will look bent. It has to be bent to look straight. See how to make it look straight. Sketch the result. A light beam passing by the Sun is bent toward the Sun. Sketch how it bends and show where the source appears to be to an observer on the Earth.

Imagine that a string has somehow been extended from our galaxy to another one two billion light years away. Set aside the difficulty that the two galaxies are receding so rapidly that a pull on one end (as would be necessary to adjust the string to the shortest route) can never reach the other (the "pull" cannot travel faster than the speed of sound).

Now sight along the string to see if it is straight. How long will that take? Where will the other galaxy be when you get through?

These questions still do not get to the bottom of the problem, but they should be sufficient to show that there are some interesting puzzles concerning the geometry of the universe.